# APMTH-105 Notes Section #9

Matheus C. Fernandes

April 13, 2015

## Goals for the week

1. Learn how to solve the diffusion equation with inhomogeneous boundary conditions.
2. Learn how to deal with the diffusion equation in polar/spherical coordinates.
3. Learn to solve the wave equation.

## Problem 1: Violin Problem Wave Equation Example

*From: AM 105 Old P-Sets*

Consider a vibrating violin spring, but with damping included. The equation for the displacement $u(x,t)$ of the string is

$$\partial_{tt}u + \zeta\partial_t u = c^2\partial_{xx}u$$

Here $\zeta$ is a damping coefficient.

(a) Solve this equation subject to the boundary and initial conditions $u(0,t) = u(L,t) = 0.$, and $u(x,t=0) = f(x)$ and $\partial_t u(x,t=0) = 0$.

(b) Give a simple estimate for how long it takes for the violin string to stop vibrating. Your estimate should be in terms of the parameters of the problem $\zeta, c, L$.

## Solution: (a)

This equation looks very similar to equations we've seen before, so let's try separation of variables. Thinking about this intuitively, we know that we want oscillatory solutions because this is a wave equation. We can also expect to find some sort of decaying exponential in the time-dependent part of the solution, since the damping will cause the oscillations to decrease with time. We begin by writing our ansatz:

$$u(x,t) = f(x)g(t). \tag{1}$$

This gives, as usual, that

$$\frac{1}{c^2}\left(\frac{g''}{g} + \zeta\frac{g'}{g}\right) = \frac{f''}{f} \equiv -k^2, \tag{2}$$

where we have chosen a negative separation constant $(-k^2)$ because we want an eigenvalue problem for our $f(x)$. Solving first the space part of the equation gets us to

$$f(x) = A\sin kx + B\cos kx, \tag{3}$$

Plugging in our boundary conditions, we see that $B\cos kx = 0$, so $B = 0$, and that $A\sin kL = 0$, so $k = n\pi/L$. Now let's consider the time-dependent solution.

$$g'' + \zeta g' + c^2k^2g = 0, \tag{4}$$

This is a second-order ODE with constant coefficients, so we try the guess $e^{\alpha t}$. Plugging this in yields the equation

$$\begin{aligned} \alpha^2 + \zeta\alpha + c^2k^2 &= 0 \\ \Rightarrow \alpha = \alpha^{\pm} &= \frac{-\zeta \pm \sqrt{\zeta^2 - 4c^2k^2}}{2}. \end{aligned} \tag{5}$$

Now, if $\zeta > 2ck$, then both $\alpha^{\pm}$ are real and negative; the solution decays monotonically in time. In contrast, if $\zeta < 2ck = 2c\pi n/L$, then the solutions are oscillatory. Now we know that the general solution is going to be a sum over all such solutions – namely we have

$$u(x,t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi x}{L}) \left[ A_n e^{\alpha_n^+ t} + B_n e^{\alpha_n^- t} \right]. \tag{6}$$

Note that if $\alpha^{\pm}$ are imaginary, then the terms in the square brackets can be rewritten, as we have before, with sine's and cosine's. Let's just ignore this for the moment–and proceed as we normally would, keeping in mind that most of the coefficients are going to be imaginary (since this occurs for all modes when $n$ is large enough). We thus write

$$u(x,t=0) = f(x) = \sum_{n=1}^{\infty} \sin(\frac{n\pi x}{L})(A_n + B_n) \tag{7}$$

whereas

$$\partial_t u(x,t=0) = 0 = \sum_{n=1}^{\infty} \sin(\frac{n\pi x}{L})(A_n\alpha_n^+ + \alpha_n^- B_n). \tag{8}$$

Orthogonality from the second equation implies that

$$A_n\alpha_n^+ + \alpha_n^- B_n = 0,$$

whereas the first equation implies that

$$A_n + B_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\ dx.$$

These are two equations and two unknowns, and so we can directly solve for $A_n$ and $B_n$. From the first equation, we have $B_n = -\alpha_n^+/\alpha_n^- A_n$, so the second equation states

$$A_n = \frac{\alpha_n^-}{\alpha_n^- - \alpha_n^+}\frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)\ dx.$$

This therefore gives the complete solution. Note that in the cases where $A_n$ is imaginary (because the $\alpha$'s are imaginary), it is necessary to use Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ to write out the solution.

**Note:** If you were to write out the solutions separately for the two cases, we see that for $\zeta > 2ck$, after we apply the initial conditions, we get:

$$u = \sum_{n=1}^{\infty} \sin kx\, e^{-\zeta t/2}\left( C_n e^{t/2\sqrt{\zeta^2-4c^2k^2}} + C_n e^{-t/2\sqrt{\zeta^2-4c^2k^2}} \right), \tag{9}$$

which can be rewritten as:

$$u = \sum_{n=1}^{\infty} \sin kx\, e^{-\zeta t/2}\left( C_n \cosh \frac{1}{2} t\sqrt{\zeta^2 - 4c^2k^2} \right). \tag{10}$$

For $\zeta < 2ck$, we get:

$$u = \sum_{n=1}^{\infty} \sin kx e^{-\zeta t/2} \left( E_n \sin(t/2\sqrt{\zeta^2 - 4c^2k^2}) + F_n \cos(t/2\sqrt{\zeta^2 - 4c^2k^2}) \right), \tag{11}$$

Now, our initial conditions give

$$\begin{aligned} u(t=0) &= \sum_{n=0}^{\infty} F_n \sin k_n x = f(x), \\ u_t(t=0) &= \sin k_n x \frac{1}{2} \left( -\zeta F_n + \sqrt{\zeta^2 - 4c^2k^2} E_n \right) = 0, \end{aligned} \tag{12}$$

so $E_n = \frac{-\zeta}{\sqrt{\zeta^2 - 4c^2k^2}} F_n$, and $F_n$ is known from a Fourier series expansion,

$$F_n = \frac{\int_0^L \sin(k_n x) f(x) \mathrm{d}x}{\int_0^L \sin^2(k_n x) \mathrm{d}x} \tag{13}$$

where the $k_n = n\pi/L$. Thus, our final solution is

$$u = \sum_{n=1}^{\infty} \sin(k_n x)\, e^{-\zeta t/2} \left( E_n \sin(t/2\sqrt{\zeta^2 - 4c^2k^2}) + F_n \cos(t/2\sqrt{\zeta^2 - 4c^2k_n^2}) \right), \tag{14}$$

where the quantities under the square roots are always positive.

## Solution: (b)

If we consider the form of our solution from part (a), notice that if we put all of the time dependence (from either case, whether $\zeta < 2ck$ or $\zeta > 2ck$) into one exponential, we get something proportional to $\exp t \left( -\zeta/2 + \sqrt{\zeta^2 - 4c^2k_n^2} \right)$. So, whenever $t \sim 1/\left( -\zeta/2 + \sqrt{\zeta^2 - 4c^2k_n^2} \right)$, then the amplitude of oscillation has gone down $\sim e^{-1}$. This is a common characterization of a time constant for exponential decay. For any given frequency ($n$), if the damping is strong enough, the square root will be positive (then you get overdamped exponential decay) and the time constant becomes smaller than $\zeta/2$. If the damping is weak, the square root will become imaginary (then you get oscillatory sines and cosines) and the time constant will be the real part of $1/\left( -\zeta/2 + \sqrt{\zeta^2 - 4c^2k_n^2} \right)$.

## Problem 2: Bug population Diffusion Equation Problem

*From: AM 105 Old P-Sets*

Consider a population of bugs. The bugs divide, and they also migrate randomly over the region that they occupy. At the borders of the region, there are terrible bug-eating monsters that eat the bugs.
These words correspond to the following mathematics problem: Let $n(x, y, t)$ be the number density of bugs. The bugs obey the equation

$$\partial_t n = D\nabla^2 n + \alpha n, \tag{15}$$

where $D$ is the diffusion constant of the bugs and $\alpha$ is the growth rate. Let's suppose the bugs are confined to a square domain, with $0 \leq x \leq L$ and $0 \leq y \leq L$. At the edge of the domain, $n(x = 0, t) = n(x = L, t) = n(y = 0, t) = n(y = L, t) = 0$. Suppose that the initial distribution of bugs is $n(x, y, t = 0) = n_0(x, y)$.

(a) Use separation of variables to find the time evolution of the number density of bugs.

(b) From your solution, answer the following question: Under what condition does the number of bugs grow in time? Under what condition does it shrink? You should invent a simple condition that depends only on the parameters $\alpha, D, L$.

## Solution: (a)

As usual, we use the ansatz $n(x, y, t) = X(x)Y(x)T(t)$. Plug this into the differential equation and divide by $DX(x)Y(y)Z(z)$ to get:

$$\frac{T'}{DT} = \frac{X''}{X} + \frac{Y''}{Y} + \frac{\alpha}{D}.$$

Since the LHS is only dependent on $t$, and the RHS is only dependent on $(x, y)$, then each side must be a constant. Furthermore, if the RHS is a constant, and $\frac{X''}{X}$ is a function of $x$ while $\frac{Y''}{Y}$ is a function of $x$, then each of these terms must also be constant. Then let

$$\begin{aligned} X''/X &= -k^2 \\ Y''/Y &= -l^2 \\ \Rightarrow T'/DT &= -k^2 - l^2 + \alpha/D \end{aligned} \tag{16}$$

Now we have three ordinary differential equations:

$$X'' + k^2 X = 0$$

$$Y'' + l^2 Y = 0$$

$$T' = -(D(k^2 + l^2) + \alpha)T$$

whose solutions are given by:

$$\begin{aligned} X(x) &= A\cos(kx) + B\sin(kx) \\ Y(y) &= C\cos(ly) + B\sin(ly) \\ T(t) &= \exp(-D(k^2 + l^2)t + \alpha t) \end{aligned} \tag{17}$$

Since we have homogeneous boundary conditions, we don't need to consider the cases when $k, l = 0$. Boundary conditions $n(x = 0) = n(y = 0) = 0$ imply that $A = C = 0$. Boundary conditions $n(x = L) = n(y = L) = 0$ imply that $k = \frac{m\pi}{L}$ and $l = \frac{n\pi}{L}$, where $m$ and $n$ are integers. Now we have the full solution:

$$n(x,y,t) = \sum_m \sum_n A_{mn} \sin(\frac{m\pi x}{L}) \sin(\frac{n\pi y}{L}) \exp\left(-D\frac{\pi^2}{L^2}(m^2+n^2)t + \alpha t\right) \tag{18}$$

## Solution: (b)

We see that the time evolution term of the solution, $T(t)$, is characterized by the expression $-D\frac{\pi^2}{L^2}(m^2+n^2)+\alpha$. Since the time dependence occurs $\forall m,n > 0$, the dominating term time dependence will be when the expression is greatest, when $m = n = 1$. Here, the expression becomes $-D\frac{2\pi^2}{L^2}+\alpha$. The population will grow in time if this expression is positive, and shrink when it is negative. Thus, when $\frac{2D\pi^2}{L^2} > \alpha$ the population will shrink and when $\frac{2D\pi^2}{L^2} < \alpha$, the population will grow.