# APMTH-105 Notes Section #8

Matheus C. Fernandes

April 6, 2015

## Goals for the week

1. Learn how to solve Laplace equation.
2. Learn how to derive the diffusion equation starting from random walkers.
3. Learn how to solve the diffusion equation.

## Problem 1: Solving Laplace Equation for semi-infinite strip

*From: Greenberg*

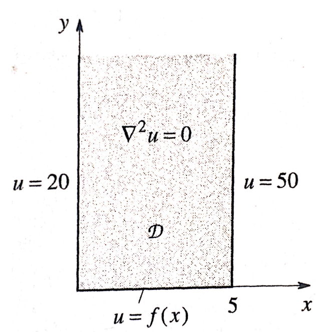


**Figure 1:** Semi-infinite strip with boundary conditions for problem 1

Consider the following Laplaces equation problem with Dirichlet boundary conditions (boundary conditions on function not derivative):

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \text{ in } \mathfrak{D} \text{ as show in Fig 1,} \tag{1}$$

$$u(0,y) = 20, u(5,y) = 50, (0 < y < \infty) \tag{2}$$

$$u(x,0) = f(x), (0 < x < 5) \tag{3}$$

$$u(x,y) \text{ bounded as } y \to \infty, \tag{4}$$

where $\mathfrak{D}$ is the semi-infinite strip $0 < x < 5, 0 < y < \infty$. As we did in class, we first seek a solution of form

$$u(x,y) = X(x)Y(y). \tag{5}$$

We anticipate that the eventual Fourier series expansion will be a half- or quarter-range expansion on the edge $y = 0$. Thus, to obtain oscillatory functions of $x$ rather than of $y$ we putting (5) into (1) and set it to be a negative constant so we get

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\kappa^2 \tag{6}$$

This yields to the following two ODE'S

$$X'' + \kappa^2 X = 0 \tag{7}$$

$$Y'' - \kappa^2 Y = 0 \tag{8}$$

This gives a solution

$$X(x) = \begin{cases} A + Bx & \kappa = 0 \\ C\cos(\kappa x) + D\sin(\kappa x) & \kappa \neq 0 \end{cases} \tag{9}$$

$$Y(x) = \begin{cases} E + Fy & \kappa = 0 \\ Ge^{\kappa y} + He^{-\kappa y} & \kappa \neq 0 \end{cases} \tag{10}$$

Because Laplaces equation is linear, we can use superposition and combine the $\kappa = 0$ and $\kappa \neq 0$ product solution as

$$u(x, y) = (A + Bx)(E + Fy) + (C\cos(\kappa x) + D\sin(\kappa x)(Ge^{\kappa y} + He^{-\kappa y}) \tag{11}$$

We apply the boundedness condition on first. Since $y$ and $e^{\kappa y}$ terms in (11) grow unboundedly as $y \to \infty$, we must set $F = 0$ and $G = 0$ to eliminate those terms. Then (11) becomes

$$u(x, y) = I + Jx + (P\cos(\kappa x) + Q\sin(\kappa x))e^{-\kappa y} \tag{12}$$

where we combined some coefficients into new coefficients. Since we anticipate the Fourier expansion to be on the $y = 0$ edge, we save that boundary condition for last. Next in line is

$$u(0, y) = 20 = I + Pe^{-\kappa y}. \tag{13}$$

Matching the coefficients of the constant and $e^{-\kappa y}$ terms on both sides gives $I = 20$ and $P = 0$. Using this to update (12) we get

$$u(x, y) = 20 + Jx + +Q\sin(\kappa x)e^{-\kappa y}. \tag{14}$$

Next,

$$u(5, y) = 50 = 20 + J5 + Q\sin(\kappa 5)e^{-\kappa y} \tag{15}$$

so $50 = 20 + 5J$ and $\sin(5\kappa) = 0$. Thus, $J = 6$ and $\kappa = n\pi/5 (n = 1, 2, 3 ...)$. Putting these results together we get with the help of superposition

$$u(x, y) = 20 + 6x + \sum_{n=1}^{\infty} Q_n \sin\frac{n\pi x}{5} e^{-n\pi y/5} \tag{16}$$

Finally we must solve

$$u(x, 0) = f(x) = 20 + 6x + \sum_{n=1}^{\infty} Q_n \sin\frac{n\pi x}{5}. \tag{17}$$

Moving things around we get that the solution is

$$f(x) - 20 - 6x = \sum_{n=1}^{\infty} Q_n \sin\frac{n\pi x}{5}. \qquad (0 < x < 5) \tag{18}$$

where we can use the half-range sine expansion of $f(x) - 20 - 6x$, so we can compute $Q_n$'s from

$$Q_n = \frac{2}{5}\int_0^5 [f(x) - 20 - 6x] \sin\frac{n\pi x}{5} dx \tag{19}$$

where to find the final solution we plug (19) into (16).

## Problem 2: Solving Diffusion Equation By Using Greens Function

Consider the diffusion equation

$$\partial_t n = D\partial_{xx} n, \tag{20}$$

with the initial condition $n(x,0) = 100$ when $-5 \leq x \leq -4$ and $n(x,0) = 200$ when $4 \leq x \leq 5$.

1. Using the Green's function for the diffusion equation, write down the solution to this equation $n(x,t)$. Your solution will be in terms of integrals that you probably will not be able to evaluate.

2. However, you should be able to write down a simple formula for the behavior of the solution at sufficiently long times. What is this solution? Roughly how long do you have to wait for the solution to be well described by your formula?

## Solution: 1.

Using the Green's Function for the diffusion equation, we have:

$$n(x,t) = \int_{-\infty}^{\infty} n(x,0) G(x - x', t)\, dx' \tag{21}$$

$$n(x,t) = 100\int_{-5}^{-4} G(x - x', t)\, dx' + 200\int_{4}^{5} G(x - x', t)\, dx' \tag{22}$$

## Solution: 2.

After long times, the initial conditions of the system won't matter much. Therefore, it's reasonable to approximate long times as two delta functions at $t = 0$. We want to conserve the number of particles for our approximation, which is straightforward to do with delta functions, via the property

$$\int_{-\infty}^{\infty} f(x)\delta(x' - x)\, dx' = f(x)$$

which gives

$$\int_{-5}^{-4} 100\, dx' = 100$$

$$\int_{4}^{5} 200\, dx' = 200$$

As such, we can replace the initial conditions at long times with

$$f(x,0) = 100\delta(x + \frac{9}{2}) + 200\delta(x - \frac{9}{2})$$

And the previous integral in (21) becomes

$$\int_{-\infty}^{\infty} n(x,0)G(x-x',t)\,dx' = \int_{-\infty}^{\infty} [100\delta(x+\frac{9}{2}) + 200\delta(x-\frac{9}{2})]G(x-x',t)\,dx'$$

$$\int_{-\infty}^{\infty} n(x,0)G(x-x',t)\,dx' = 100G(-\frac{9}{2}-x,t) + 200G(\frac{9}{2}-x,t)$$

And so $n(x,t)$ can be approximated at long times by:

$$\boxed{n(x,t) \approx \frac{100}{\sqrt{4\pi Dt}}(e^{-(x+9/2)^2/(4Dt)} + 200e^{-(x-9/2)^2/(4Dt)})}$$

*How long do you have to wait until this approximation is close to the real system?* We can think of the initial distribution as a sum of delta functions, weighted so that the total number of particles is conserved. We may consider only the worst-case scenario: the delta functions that lie at $x=-3$ and at $x=-2$. We know the solution for a delta function spreads like $\sqrt{4Dt}$, and the approximation becomes appropriate when this length ($\sqrt{4Dt}$)) is larger than the length scale of the initial condition. If we consider each of the spikes in the initial condition separately, they each have a width =1 and so we expect the approximation to hold when $\sqrt{4Dt} > 1$, or $\boxed{t > 1/(4D)}$. On the other hand, we can also look at this problem differently. At even longer times,memory of the initial condition will be completely lost. The solution will be the same as if ALL of the mass of the solution were concentrated at the origin. At that point

$$n(x,t) \approx \frac{300}{\sqrt{4\pi Dt}}(e^{-(x)^2/(4Dt)}).$$

This is the very long time solution. It will occur when each of the spreading gaussian's in the solution above spread out enough to overlap. Since they are a distance $\sim 9$ apart, this will happen when $\sqrt{4Dt} \sim 9$, or when $\boxed{t > 81/(4D)}$.