APMTH-105 Notes Section #2

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Goals for the week

1. Learn and apply (MATLAB) numerical methods to solve ODEs.
2. Learn some methods of constructing exact solutions for linear second-order ODEs.
3. Refresh linear dependence/independence.
4. Refresh how to find approximate solutions for ODEs using dominant balance.

Problem 1: Homogeneous Linear Second Order Equation

Solve for the initial value problem

$$y'' + 2y' - y = 0 \text{ for } y(0) = 0\,, y'(0) = -1 \tag{1}$$

We will first find a pair of solutions using the auxiliary equation $y = e^{rt}$ such that we have

$$r^2 + 2r - 1 = 0 \tag{2}$$

Using the quadratic formula, we find that the roots of this equation are:

$$r_1 = -1 + \sqrt{2} \text{ and } r_2 = -1 - \sqrt{2} \tag{3}$$

Consequently, the given differential equation has solutions of the form:

$$y(t) = c_1 e^{(-1+\sqrt{2})t} + c^2 e^{(-1-\sqrt{2})t} \tag{4}$$

To find the specific solution that satisfies the initial conditions given, we firs differential y as giben in the general solutionm then plug y and y' into the initial conditions. This gives

$$y(0) = c_1 e^0 + c_2 e^0 \tag{5}$$

$$y'(0) = \left(-1 + \sqrt{2}\right) c_1 e^0 + \left(-1 - \sqrt{2}\right) c_2 e^0 \tag{6}$$

or

$$0 = c_1 + c_2 \tag{7}$$

$$-1 = \left(-1 + \sqrt{2}\right) c_1 + \left(-1 - \sqrt{2}\right) c_2 \tag{8}$$

Solving the system of equations yields $c_1 = -\sqrt{2}/4$ and $c_2 = \sqrt{2}/4$. Thus,

$$\boxed{y(t) = -\frac{\sqrt{2}}{4} e^{(-1+\sqrt{2})t} + \frac{\sqrt{2}}{4} e^{(-1-\sqrt{2})t}} \tag{9}$$

Problem 2: Euler Equations

Solve for the initial value problem

$$2x^2y'' + 3xy' - 15y = 0 \text{ for } y(1) = 0\,,\, y'(1) = 1 \tag{10}$$

Let's first start from the general euler equation

$$x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0. \tag{11}$$

Using the ansatz: $y(x) = x^m$, difrrentiating we have:

$$\frac{dy}{dx} = mx^{m-1} \tag{12}$$

$$\frac{d^2y}{dx^2} = m(m-1)x^{m-2} \tag{13}$$

We can plug this into the original differential equation:

$$x^2(m(m-1)x^{m-2}) + ax(mx^{m-1}) + b(x^m) = 0 \tag{14}$$

And rearranging gives

$$m^2 + (a-1)m + b = 0 \tag{15}$$

Dividing (10) by 2 we get:

$$x^2y'' + \frac{3}{2}xy' - \frac{15}{2}y = 0 \tag{16}$$

Where it is now of from noted in (11), so we can solve the auxiliary equation described in (15) which yields:

$$m^2 + (\frac{3}{2} - 1)m - \frac{15}{2} = 0 \tag{17}$$

$$m^2 + \frac{1}{2}m - \frac{15}{2} = 0 \tag{18}$$

$$2m^2 + m - 15 = (2m-5)(m+3) = 0 \tag{19}$$

So our roots then become:

$$r_1 = \frac{5}{2}\,, \text{ and } r_2 = -3 \tag{20}$$

The general solution is then,

$$y(x) = c_1x^{5/2} - c_2x^{-3} \tag{21}$$

We now need to find the constants using the initial conditions given in (10):

$$0 = y(1) = c_1 + c_2 \tag{22}$$

$$1 = y'(1) = \frac{5}{2}c_1 - 3c_2 \tag{23}$$

Solving the system of equations we have:

$$c_1 = \frac{2}{11}\,,\, c_2 = -\frac{2}{11} \tag{24}$$

So the actual solution becomes:

$$\boxed{y(x) = \frac{2}{11}x^{5/2} - \frac{2}{11}x^{-3}} \tag{25}$$

Problem 3: Linearly Dependent of Independent

From: From Paul's Online notes at Lamar University, see attached.

Determine id the following functions are linearly dependent or linearly independent:

$$f(x) = 9\cos(2x) \tag{26}$$

$$g(x) = 2\cos^2(x) - 2\sin^2(x) \tag{27}$$

Lets write down in terms of the following equation:

$$cf(x) + kg(x) = 0 \tag{28}$$

So

$$c(9\cos(2x)) + k(2\cos^2(x) - 2\sin^2(x)) = 0 \tag{29}$$

With this simplification we can see that this will be zero for any pair of constants c and k that satisfy

$$9c + 2k = 0 \tag{30}$$

Among the possible pairs on constants that we could use are the following pairs.

$$c = 1, \qquad k = -\frac{9}{2} \tag{31}$$

$$c = \frac{2}{9}, \qquad k = -1 \tag{32}$$

$$c = -2, \qquad k = 9 \tag{33}$$

$$c = -\frac{7}{6}, \qquad k = \frac{21}{4} \tag{34}$$

So we have managed to find a pair of non-zero constants that will make the equation true for all x and so the two functions are **linearly dependent**.

Problem 4: Numerical Methods Euler's Method

Suppose we want to numerically solve the following Bernoulli equation which we know the analytical solution to:

$$\frac{dy}{dx} = 5y - \frac{5}{2}xy^3 \text{ with } y(0) = 1 \tag{35}$$

is

$$y(x) = \frac{2\sqrt{5}e^{5x}}{\sqrt{e^{10x}(10x-1)+21}} \tag{36}$$

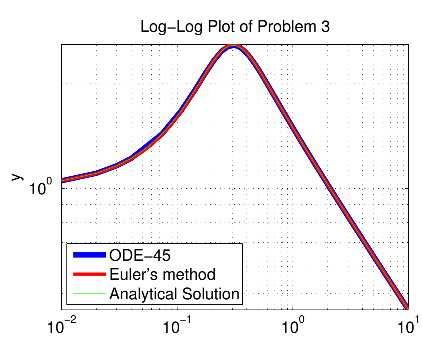

Figure 1: Plot comparing ODE45, Analytical and Euler's Method for $h = 0.01$

Code for Problem 4 graph

```
%Harvard Univeristy - AM105 Example
%By: Matheus Fernandes
clc
close all
clear all

dydx= @(x,y) 5*y-5/2*x*y^3;
[xx,yy] = ode45(dydx,[0 10],1);

h=0.0001; % stepsize
x0=0;
xfinal=10; % final position

x=x0:h:xfinal;

y(1)=1;  % initial condition
for i=1:length(x)-1

    y(i+1)=y(i)+h*dydx(x(i),y(i));
end


y(end)  % value at x=xfinal for Euler's method
yy(end) % value at x=xfinal for "exact solution" (We take MATLAB's ODE solver as the exact solution)

analytical= (2.*sqrt(5).*exp(5*x))./sqrt(exp(10.*x).*(10.*x-1)+21);

figure(1)
loglog(xx,yy,'b','linewidth',5); hold on
loglog(x,y,'r','linewidth',3)
loglog(x,analytical,'g')
xlim([0.01 10])

title('Log-Log Plot of Problem 3','fontsize',15)
xlabel('x','fontsize',15)
ylabel('y','fontsize',15)
legend('ODE-45','Euler''s method','Analytical Solution','location','southwest')
set(gca,'fontsize',15)
grid on
```

More on the Wronskian

In the previous section we introduced the Wronskian to help us determine whether two solutions were a fundamental set of solutions. In this section we will look at another application of the Wronskian as well as an alternate method of computing the Wronskian.

Let's start with the application. We need to introduce a couple of new concepts first.

Given two non-zero functions *f(x)* and *g(x)* write down the following equation.

$$c\, f\left(x\right)+k\, g\left(x\right)=0 \qquad \textbf{(1)}$$

Notice that $c = 0$ and $k = 0$ will make (1) true for all x regardless of the functions that we use.

Now, if we can find non-zero constants c and k for which (1) will also be true for all x then we call the two functions **linearly dependent**. On the other hand if the only two constants for which (1) is true are $c = 0$ and $k = 0$ then we call the functions **linearly independent.**

Example 1 Determine if the following sets of functions are linearly dependent or linearly independent.

(a) $f\left(x\right)=9\cos\left(2x\right) \qquad g\left(x\right)=2\cos^2\left(x\right)-2\sin^2\left(x\right)$ [Solution]

(b) $f\left(t\right)=2t^2 \qquad g\left(t\right)=t^4$ [Solution]

Solution

(a) $f\left(x\right)=9\cos\left(2x\right) \qquad g\left(x\right)=2\cos^2\left(x\right)-2\sin^2\left(x\right)$

We'll start by writing down (1) for these two functions.

$$c\left(9\cos\left(2x\right)\right)+k\left(2\cos^2\left(x\right)-2\sin^2\left(x\right)\right)=0$$

We need to determine if we can find non-zero constants c and k that will make this true for all x or if $c = 0$ and $k = 0$ are the only constants that will make this true for all x. This is often a fairly difficult process. The process can be simplified with a good intuition for this kind of thing, but that's hard to come by, especially if you haven't done many of these kinds of problems.

In this case the problem can be simplified by recalling

$$\cos^2\left(x\right)-\sin^2\left(x\right)=\cos\left(2x\right)$$

Using this fact our equation becomes.

$$\begin{aligned} 9c\cos\left(2x\right)+2k\cos\left(2x\right)&=0 \\ \left(9c+2k\right)\cos\left(2x\right)&=0 \end{aligned}$$

With this simplification we can see that this will be zero for any pair of constants c and k that satisfy

$$9c+2k=0$$

Among the possible pairs on constants that we could use are the following pairs.

$$
\begin{aligned}
c &= 1, & k &= -\frac{9}{2} \\
c &= \frac{2}{9}, & k &= -1 \\
c &= -2 & k &= 9 \\
c &= -\frac{7}{6} & k &= \frac{21}{4} \\
etc.
\end{aligned}
$$

As I'm sure you can see there are literally thousands of possible pairs and they can be made as "simple" or as "complicated" as you want them to be.

So, we've managed to find a pair of non-zero constants that will make the equation true for all x and so the two functions are linearly dependent.

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(b) $f(t) = 2t^2 \qquad g(t) = t^4$

As with the last part, we'll start by writing down (1) for these functions.

$$2ct^2 + kt^4 = 0$$

In this case there isn't any quick and simple formula to write one of the functions in terms of the other as we did in the first part. So, we're just going to have to see if we can find constants. We'll start by noticing that if the original equation is true, then if we differentiate everything we get a new equation that must also be true. In other words, we've got the following system of two equations in two unknowns.

$$
\begin{aligned}
2ct^2 + kt^4 &= 0 \\
4ct + 4kt^3 &= 0
\end{aligned}
$$

We can solve this system for c and k and see what we get. We'll start by solving the second equation for c.

$$c = -kt^2$$

Now, plug this into the first equation.

$$
\begin{aligned}
2\left(-kt^2\right)t^2 + kt^4 &= 0 \\
-kt^4 &= 0
\end{aligned}
$$

Recall that we are after constants that will make this true for all t. The only way that this will ever be zero for all t is if $k = 0$! So, if $k = 0$ we must also have $c = 0$.

Therefore, we've shown that the only way that

$$2ct^2 + kt^4 = 0$$

will be true for all t is to require that $c = 0$ and $k = 0$. The two functions therefore, are linearly independent.

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As we saw in the previous examples determining whether two functions are linearly independent or linearly dependent can be a fairly involved process. This is where the Wronskian can help.

Fact

Given two functions *f(x)* and *g(x)* that are differentiable on some interval I.

(1) If $W\left(f,g\right)\left(x_0\right) \neq 0$ for some x_0 in I, then *f(x)* and *g(x)* are linearly independent on the interval I.

(2) If *f(x)* and *g(x)* are linearly dependent on I then *W(f,g)(x)* = 0 for all *x* in the interval I.

Be very careful with this fact. It DOES NOT say that if *W(f,g)(x)* = 0 then *f(x)* and *g(x)* are linearly dependent! In fact it is possible for two linearly independent functions to have a zero Wronskian!

This fact is used to quickly identify linearly independent functions and functions that are liable to be linearly dependent.

Example 2 Verify the fact using the functions from the previous example.

Solution

(a) $f\left(x\right) = 9\cos\left(2x\right) \qquad g\left(x\right) = 2\cos^2\left(x\right) - 2\sin^2\left(x\right)$

In this case if we compute the Wronskian of the two functions we should get zero since we have already determined that these functions are linearly dependent.

$$\begin{aligned} W &= \begin{vmatrix} 9\cos\left(2x\right) & 2\cos^2\left(x\right) - 2\sin^2\left(x\right) \\ -18\sin\left(2x\right) & -4\cos\left(x\right)\sin\left(x\right) - 4\sin\left(x\right)\cos\left(x\right) \end{vmatrix} \\ &= \begin{vmatrix} 9\cos\left(2x\right) & 2\cos\left(2x\right) \\ -18\sin\left(2x\right) & -2\sin\left(2x\right) - 2\sin\left(2x\right) \end{vmatrix} \\ &= \begin{vmatrix} 9\cos\left(2x\right) & 2\cos\left(2x\right) \\ -18\sin\left(2x\right) & -4\sin\left(2x\right) \end{vmatrix} \\ &= -36\cos\left(2x\right)\sin\left(2x\right) - \left(-36\cos\left(2x\right)\sin\left(2x\right)\right) = 0 \end{aligned}$$

So, we get zero as we should have. Notice the heavy use of trig formulas to simplify the work!

(b) $f\left(t\right) = 2t^2 \qquad g\left(t\right) = t^4$

Here we know that the two functions are linearly independent and so we should get a non-zero Wronskian.

$$W = \begin{vmatrix} 2t^2 & t^4 \\ 4t & 4t^3 \end{vmatrix} = 8t^5 - 4t^5 = 4t^5$$

The Wronskian is non-zero as we expected provided $t \neq 0$. This is not a problem. As long as the Wronskian is not identically zero for all *t* we are okay.

Example 3 Determine if the following functions are linearly dependent or linearly independent.

(a) $f(t)=\cos t \qquad g(t)=\sin t$ [Solution]

(b) $f(x)=6^x \qquad g(x)=6^{x+2}$ [Solution]

Solution

(a) Now that we have the Wronskian to use here let's first check that. If its non-zero then we will know that the two functions are linearly independent and if its zero then we can be pretty sure that they are linearly dependent.

$$W=\begin{vmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{vmatrix}=\cos^2 t+\sin^2 t=1\neq 0$$

So, by the fact these two functions are linearly independent. Much easier this time around!

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(b) We'll do the same thing here as we did in the first part. Recall that

$$\left(a^x\right)'=a^x\ln a$$

Now compute the Wronskian.

$$W=\begin{vmatrix}6^x & 6^{x+2}\\ 6^x\ln 6 & 6^{x+2}\ln 6\end{vmatrix}=6^x6^{x+2}\ln 6-6^{x+2}6^x\ln 6=0$$

Now, this does not say that the two functions are linearly dependent! However, we can guess that they probably are linearly dependent. To prove that they are in fact linearly dependent we'll need to write down (1) and see if we can find non-zero c and k that will make it true for all x.

$$\begin{aligned}c\,6^x+k\,6^{x+2}&=0\\ c\,6^x+k\,6^x6^2&=0\\ c\,6^x+36k\,6^x&=0\\ (c+36k)6^x&=0\end{aligned}$$

So, it looks like we could use any constants that satisfy

$$c+36k=0$$

to make this zero for all x. In particular we could use

$$\begin{aligned}&c=36 && k=-1\\ &c=-36 && k=1\\ &c=9 && k=-\frac{1}{4}\\ &etc.\end{aligned}$$

We have non-zero constants that will make the equation true for all x. Therefore, the functions are linearly dependent.

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